Pendulum with a cylindrical blade rolling on a horizontal plane

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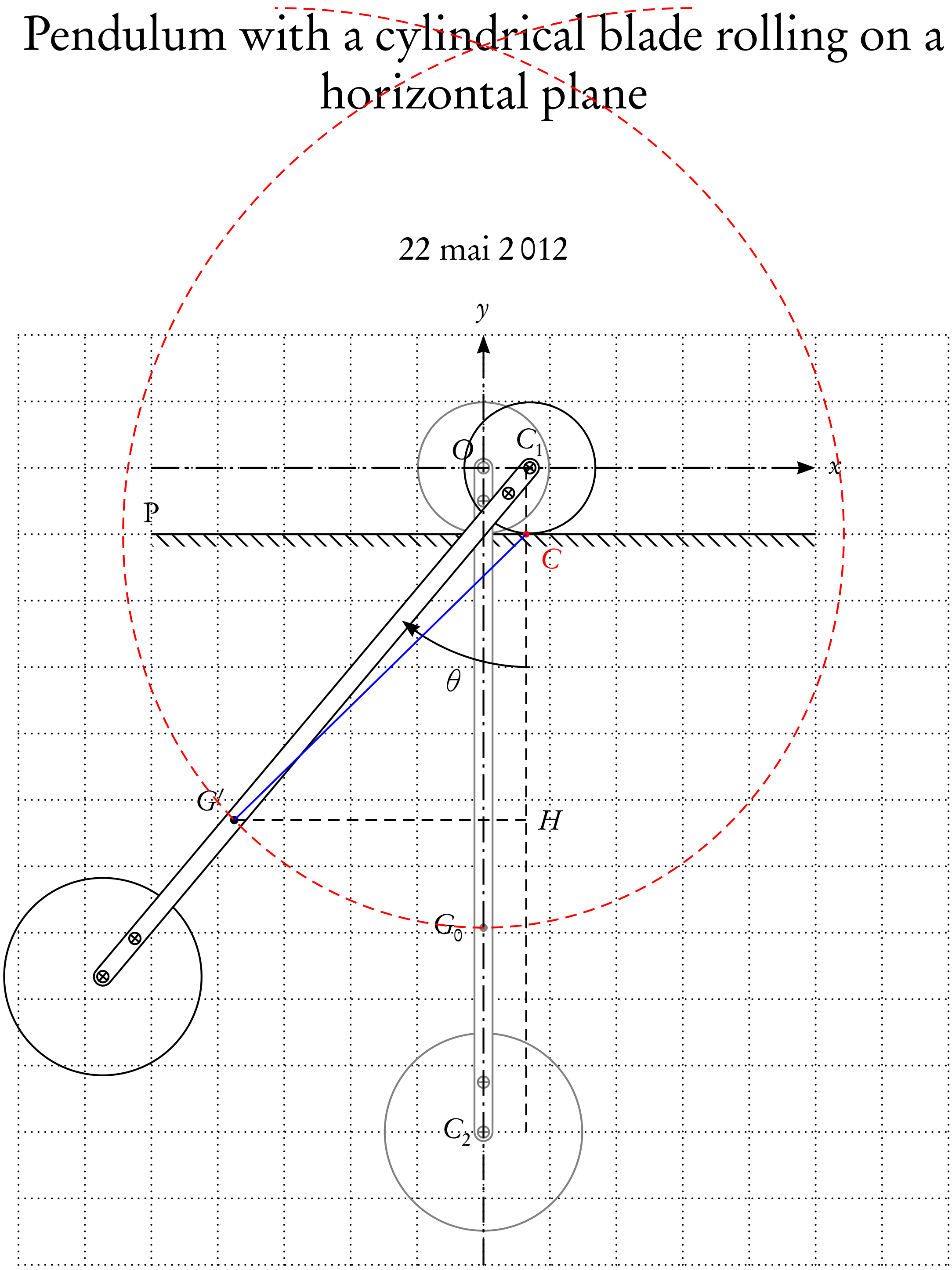


1 Equations of the motion

We find this sort of pendulum in Henri Bouasse's book "*Pendule spiral, diapason*", volume 1, pages 100, 101 and 102[1].

« Le pendule est lié à un cylindre circulaire qui roule sans glisser sur un plan P fixe horizontal.

À chaque instant le pendule admet pour axe instantané de rotation, la génératrice C de tangence du cylindre avec le plan P.»

The method Henri Bouasse follows is very interesting, cause it uses the *gyration radius*, whenever this method is quite obsolete.

The pendulum consists of a stick with negligible mass and a length ℓ. The *rolling* cylinder with the center C_1 has a radius r_1 and the mass m_1. The pendular is also a cylinder with the center C_2, radius r_2 and mass m_2. We put $\ell = C_1C_2$.

We determine the position G of the center of mass on the stick:

$$\ell_1 = C_1G = \frac{m_2\ell}{m_1+m_2}$$

The movement is without gliding, so the coordinates of G are:

$$\begin{cases} x_G &= -r_1\theta + l_1\sin\theta \\ y_G &= -l_1\cos\theta \end{cases}$$

G describes an *extended* cycloide.

The momentum of inertia with reference to G, we put ρ, the gyration radius:

$$J_G = \frac{1}{2}m_1r_1^2 + m_1l_1^2 + \frac{1}{2}m_2r_2^2 + m_2(l-l_1)^2 = (m_1+m_2)\rho^2$$

$$\rho^2 = \frac{\frac{1}{2}m_1r_1^2 + m_1l_1^2 + \frac{1}{2}m_2r_2^2 + m_2(l-l_1)^2}{m_1+m_2}$$

Let θ be the angle between the pendulum axis and the vertical at a time t, we calculate CG'.

$$CG'^2 = l_1^2 + r_1^2 - 2r_1l_1\cos\theta$$

The momentum of inertia with reference to C is:

$$J_C = (m_1+m_2)(\rho^2 + l_1^2 + r_1^2 - 2r_1l_1\cos\theta)$$

Henri Bouasse: « Nous pouvons considérer *pendant un instant* le point C comme immobile (axe instantané). Comme dans le roulement toutes les droites du solide

[1] All citations are from Henri Bouasse.

invariable tournent du même angle, la vitesse de rotation autour du point C est $\dot{\theta}$. »
Therefrom the expression for the kinetic energy:

$$\mathscr{E}_C = \frac{1}{2} J_C \dot{\theta}^2$$

The potential energy:

$$\mathscr{E}_p = -(m_1 + m_2) g l_1 \cos\theta$$

Since rolling is without gliding, we can apply the conservation of energy. The pendulum is deflected from its balanced state by an angle θ_0 and let loose without an initial velocity.[2]

$$\frac{1}{2}(m_1 + m_2)(\rho^2 + l_1^2 + r_1^2 - 2 r_1 l_1 \cos\theta)\dot{\theta}^2$$
$$-(m_1 + m_2) g l_1 \cos\theta = -(m_1 + m_2) g l_1 \cos\theta_0$$

For the calculation of the period we keep in mind, after some simplifications:

$$(m_1 + m_2)(\rho^2 + l_1^2 + r_1^2 - 2 r_1 l_1 \cos\theta) = 2(m_1 + m_2) g l_1 (\cos\theta - \cos\theta_0) \qquad (1)$$

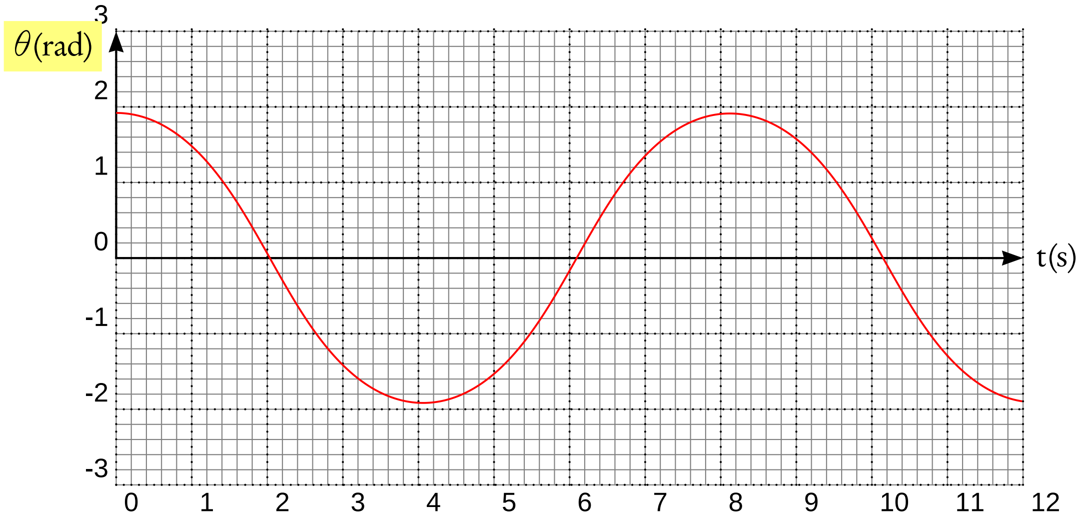
We type the derivative with respect to time, the energy is zero and we simplify:

$$0 = (\rho^2 + l_1^2 + r_1^2 - 2 r_1 l_1 \cos\theta)\ddot{\theta} + r_1 l_1 \sin\theta \dot{\theta}^2 + g l_1 \sin\theta$$

$$\ddot{\theta} = -\frac{(r_1 \dot{\theta}^2 + g) l_1 \sin\theta}{\rho^2 + l_1^2 + r_1^2 - 2 r_1 l_1 \cos\theta} \qquad (2)$$

[2] « On remarquera qu'il doit nécessairement exister une réaction tangentielle X du support. Si le frottement était nul, le pendule ne roulerait pas : il se déplacerait en bloc dès que la droite joignant le point de contact au centre d'inertie cesserait d'être verticale. Mais le mouvement étant un pur roulement, la force X ne travaille pas. »

Henri Bouasse : "*Pendule spiral, diapason*", volume 1, page 102.



2 Calculations for the period

>From equation (1), we deduce:

$$\dot{\theta} = \sqrt{\frac{2gl_1(\cos\theta - \cos\theta_0)}{\rho^2 + l_1^2 + r_1^2 - 2r_1 l_1 \cos\theta}}$$

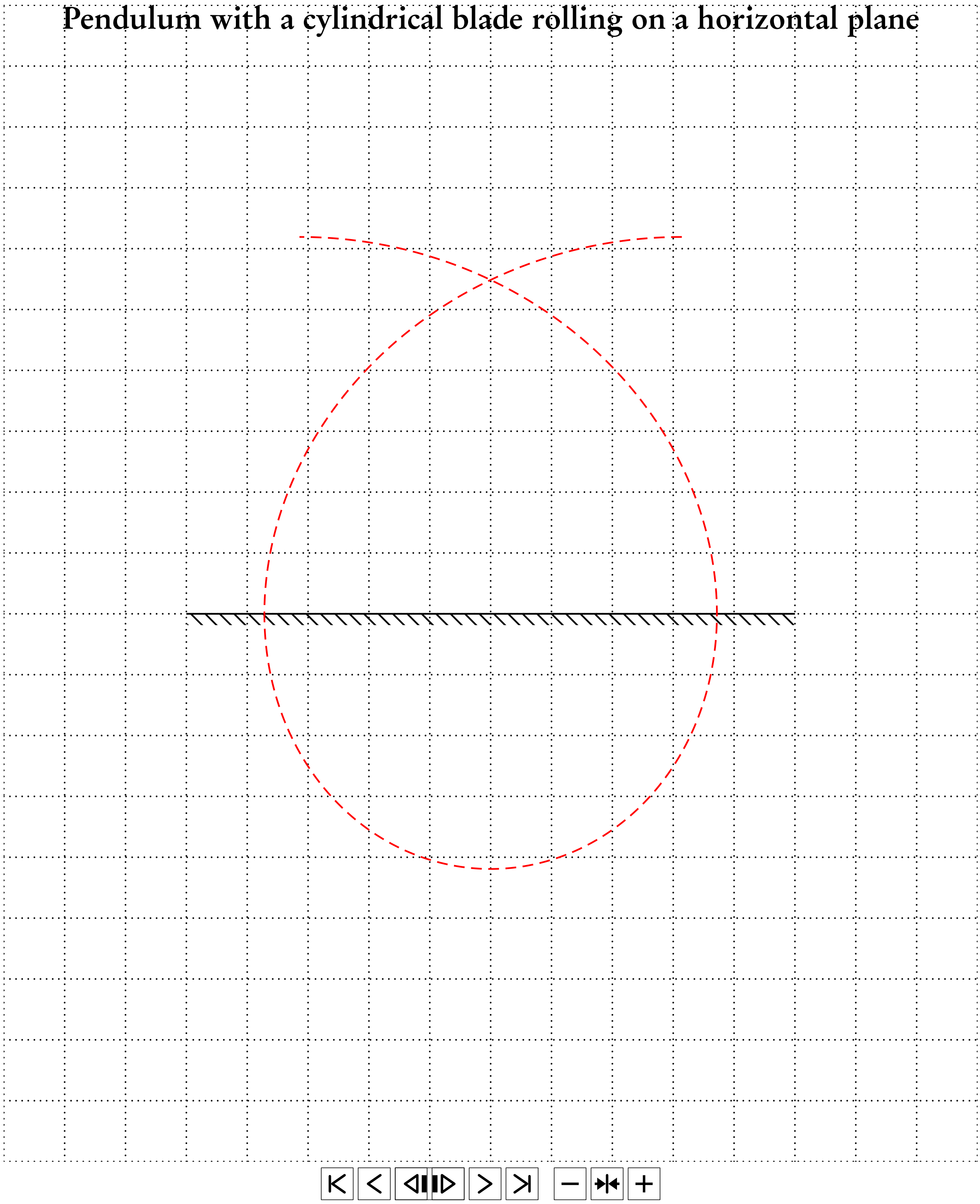
$$T = 4\int_0^{\theta_0} \sqrt{\frac{(\rho^2 + l_1^2 + r_1^2 - 2r_1 l_1 \cos\theta)}{2gl_1(\cos\theta - \cos\theta_0)}} d\theta \qquad (3)$$

T=8,10553 s

This is a quite huge period, but the pendulum studied theoretically has some gigantic dimensions!

```
/theta0 110 deg2rad def % en radians
/r1 1 def % rayon du cylindre couteau en m
/r2 1.5 def % rayon de la lentille du pendule en m
/L0 10 def % longueur du pendule en m
/m1 1 def % masse du cylindre couteau en kg
/m2 2.25 def % masse de la lentille en kg
```

3 Animation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4 The used PSTricks macros

\psRK permits the solution to the differential equation (2). It calculates a tableau of values stacked in the PostScript variable `tabTheta` : $[\ldots t_i\ \theta_i\ \ldots]$ which allows to one side to trace the line of the function $\theta = f(t)$ and on the other side to draw the pendulum to a whatever time. It also allows to simulate the oscillations in real time. `\psRK[options](borne_inf,borne_sup){fonction}`. For the precision, the step is calculated from the number of chosen points: `plotpoints`.

```
\def\thetapendule{-((r1*l1*y1^2+G*l1)*sin(y0))/(rho+l1^2+r1^2-2*r1*l1*cos(y0))}%
\pstVerb{
    /Pi 3.1415926 def
    /deg2rad {180 div Pi mul} def
    /rad2deg {180 mul Pi div} def
    /G 9.8 def
    /theta0 110 deg2rad def % en radians
    /thetapoint0 0 def
    /r1 1 def % rayon du cylindre couteau
    /r2 1.5 def % rayon de la lentille du pendule
    /L0 10 def % O1O2 du pendule
    /m1 1 def % masse du cylindre couteau
    /m2 2.25 def % masse de lentille
    /l1 m2 L0 mul m1 m2 add div def % O1G distance de O1 au centre de gravit\'{e}
    /l2 L0 l1 sub def % O2G
    /JG m1 r1 dup mul mul 0.5 mul m2 r2 dup mul mul 0.5 mul add
        m1 l1 dup mul mul m2 l2 dup mul mul add add def % J/G
    /rho JG m1 m2 add div def % rayon de giration au carr\'{e}
     }%
\psRK[algebraic,plotpoints=1000](0,12){\thetapendule}%
\listplot[linecolor=red,linewidth=0.025]{tabTheta aload pop}%
```

\psInt permits to calculate the integral, employed to calculate period of the oscillations (3). `\psInt[options](borne_inf,borne_sup){fonction}`.

```
\def\periode{4*sqrt((rho+l1^2+r1^2-2*r1*l1*cos(t))/(2*G*l1*abs((cos(t)-costheta0)))
\pstVerb{
    /Pi 3.1415926 def
    /deg2rad {180 div Pi mul} def
    /rad2deg {180 mul Pi div} def
    /G 9.8 def
    /theta0 110 deg2rad def % en radians
    /costheta0 110 cos def
```

```
    /thetai theta0 0.99999 mul def
    /thetapoint0 0 def
    /r1 1 def % rayon du cylindre couteau
    /r2 1.5 def % rayon de la lentille du pendule
    /L0 10 def % O1O2 du pendule
    /m1 1 def % masse du cylindre couteau
    /m2 2.25 def % masse de lentille
    /l1 m2 L0 mul m1 m2 add div def % O1G distance de O1 au centre de gravit\'{e}
    /l2 L0 l1 sub def % O2G
    /JG m1 r1 dup mul mul 0.5 mul m2 r2 dup mul mul 0.5 mul add
        m1 l1 dup mul mul m2 l2 dup mul mul add add def % J/G
    /rho JG m1 m2 add div def % rayon de giration au carr\'{e}
     }%
\psInt[algebraic](0,thetai){\periode}
\rput(0,0){T=\psPrintValue{I}\hphantom{000000}s}
```

5 A second method

Henri Bouasse proposes another method to find the differential equation (2) of the motion.

We remind the coordinates of G:

$$\left\{\begin{array}{lcr} x &=& -r_1\theta + l_1\sin\theta \\ y &=& -l_1\cos\theta \end{array}\right.$$

The tangential and normal components of the reaction force to the support $\overrightarrow{R}$ are named X and Y. First we apply the theorem of the center of mass (a fundamental relation of dynamic systems including, that all forces are located in the center of mass). Note: $m = m_1 + m_2$, the total mass of the pendulum.

$$m\frac{\mathrm{d}^2\overrightarrow{OG}}{\mathrm{d}t^2} = m\overrightarrow{\mathrm{g}} + \overrightarrow{R}$$

$$\left\{\begin{array}{lcr} \ddot{x} &=& \ddot{\theta}(-r_1 + l_1\cos\theta) - l_1\dot{\theta}^2\sin\theta \\ \ddot{y} &=& l_1\ddot{\theta}\sin\theta + l_1\dot{\theta}^2\cos\theta \end{array}\right.$$

With a projection onto the axes, the theorem of the center of mass gives:

$$\left\{\begin{array}{rcl} m\ddot{\theta}(-r_1 + l_1\cos\theta) - ml_1\dot{\theta}^2\sin\theta &=& X \\ ml_1\ddot{\theta}\sin\theta + ml_1\dot{\theta}^2\cos\theta &=& Y - m\mathrm{g} \end{array}\right.$$

$$\begin{cases} X = m\Big(\ddot{\theta}(-r_1 + l_1\cos\theta) - l_1\dot{\theta}^2\sin\theta\Big) \\ Y = m\Big(l_1\ddot{\theta}\sin\theta + l_1\dot{\theta}^2\cos\theta + g\Big) \end{cases}$$

In a second step, we apply the theorem for the dynamic momentum of the center of mass G:

$$m\rho^2\ddot{\theta} = -X(l_1\cos\theta - r_1) - Y l_1\sin\theta$$

Replace X and Y with their respective expressions and cancel down with m:

$$\rho^2\ddot{\theta} = -\Big(\ddot{\theta}(-r_1 + l_1\cos\theta) - l_1\dot{\theta}^2\sin\theta\Big)(l_1\cos\theta - r_1) - (l_1\ddot{\theta}\sin\theta + l_1\dot{\theta}^2\cos\theta + g)l_1\sin\theta$$

After some more simplifications, we receive the same expression for $\ddot{\theta}$:

$$\ddot{\theta} = -\frac{(r_1\dot{\theta}^2 + g)l_1\sin\theta}{\rho^2 + l_1^2 + r_1^2 - 2r_1 l_1\cos\theta}$$